



Fracture Mechanics

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The background of the slide features a large, light blue watermark of the official emblem of the Islamic Consultative Assembly of Iran. The emblem is circular, with a scalloped outer edge. Inside the circle, at the top, is a stylized representation of the word 'Allah' in Arabic calligraphy. Below this, the word 'Majlis' is written in Persian script. At the bottom of the emblem are two large, symmetrical wings, reminiscent of the wings of the Angel Gabriel. The entire emblem is set against a white background.

Energy theories

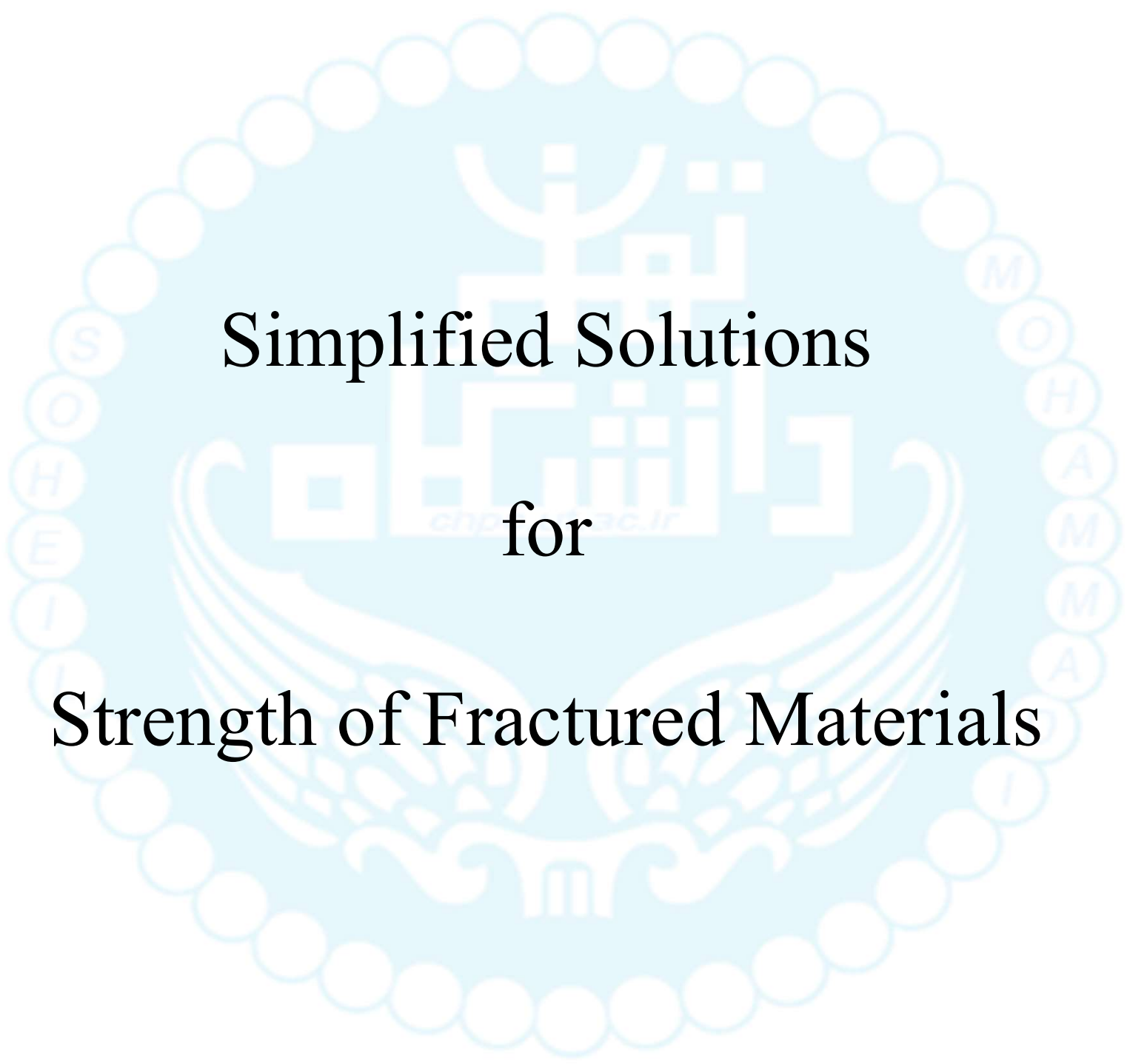
Griffith energy theories

While exploring the theoretical strength of solids by performing a series of experiments on glass rods of various diameters, Griffith (1920) observed that the tensile strength of glass decreased with an increase in diameter. He realised that something different from a simple inherent material property had caused the size dependency of the tensile strength.

While studying the solution of the elliptical hole problem, Inglis initiated the idea that the theoretical strength of a solid has to be reduced due to the presence of internal flaws. In other words, he assumed that the theoretical strength of a material must be compared with the concentrated stress field, which is much higher than the average stress or the flawless based stress field.

Original contributions were based on simplified assumptions to derive approximate solutions for material strength based on potential atomistic defects. These solutions were the first theoretical methods, though very simplified, to connect the macro related strength with nano (micro) related defects.

Instead of using a stress based criterion, Griffith derived a thermodynamic criterion for fracture by considering the total change in energy of a cracked body in terms of the crack length increase. His model described the failure of a solid material in terms of satisfying a critical energy criterion rather than a maximum stress based failure control.



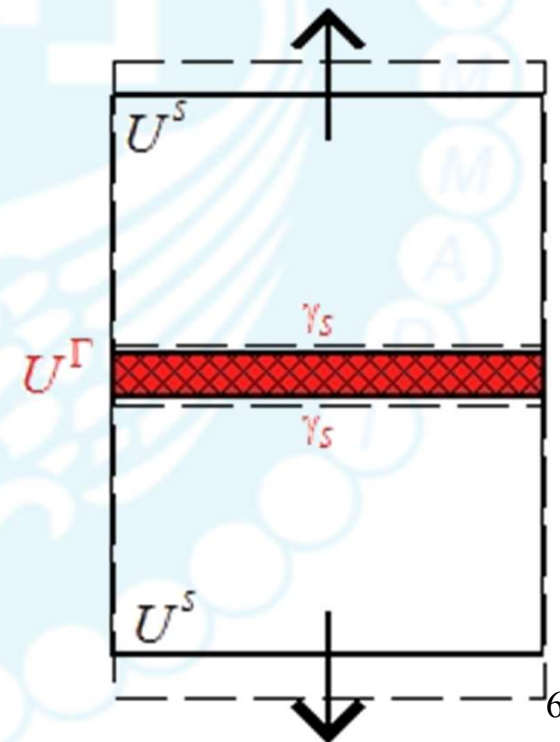
Simplified Solutions for Strength of Fractured Materials

Here, expressions are developed for the theoretical strength of perfect crystals (theoretically the strongest form of solid). This derivation, (Kelly 1974) is fundamentally different than the one of Griffith, as it starts at the atomic level.

The energy of interaction between two adjacent atoms at equilibrium separated by a distance a_0 .

The total energy which must be supplied to separate a layer of atoms from its neighbor layer

$$U^{\Gamma} = 2\gamma_s$$



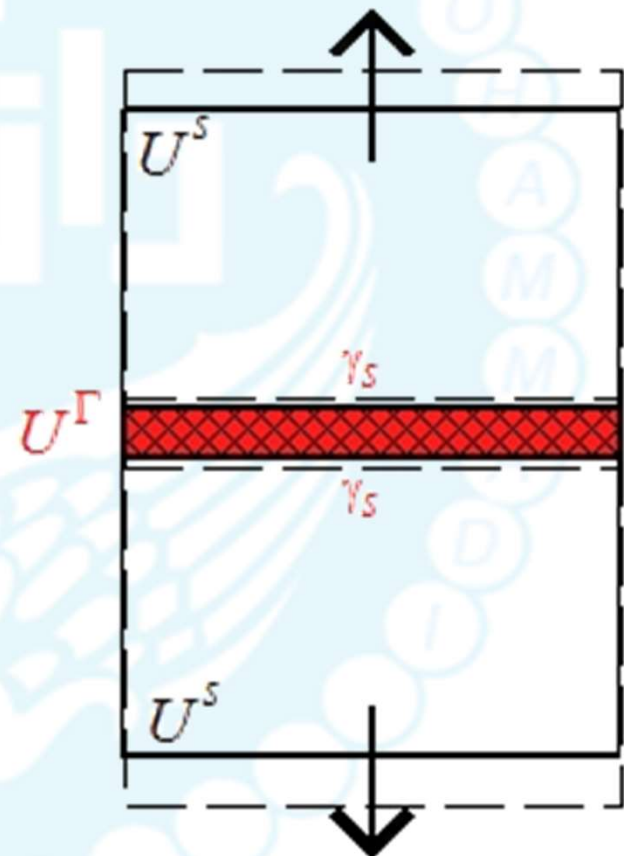
γ_s is the surface energy and the factor 2 represents the existence of two material surfaces upon fracture. The value of γ_s is experimentally measured for different materials.

Water:

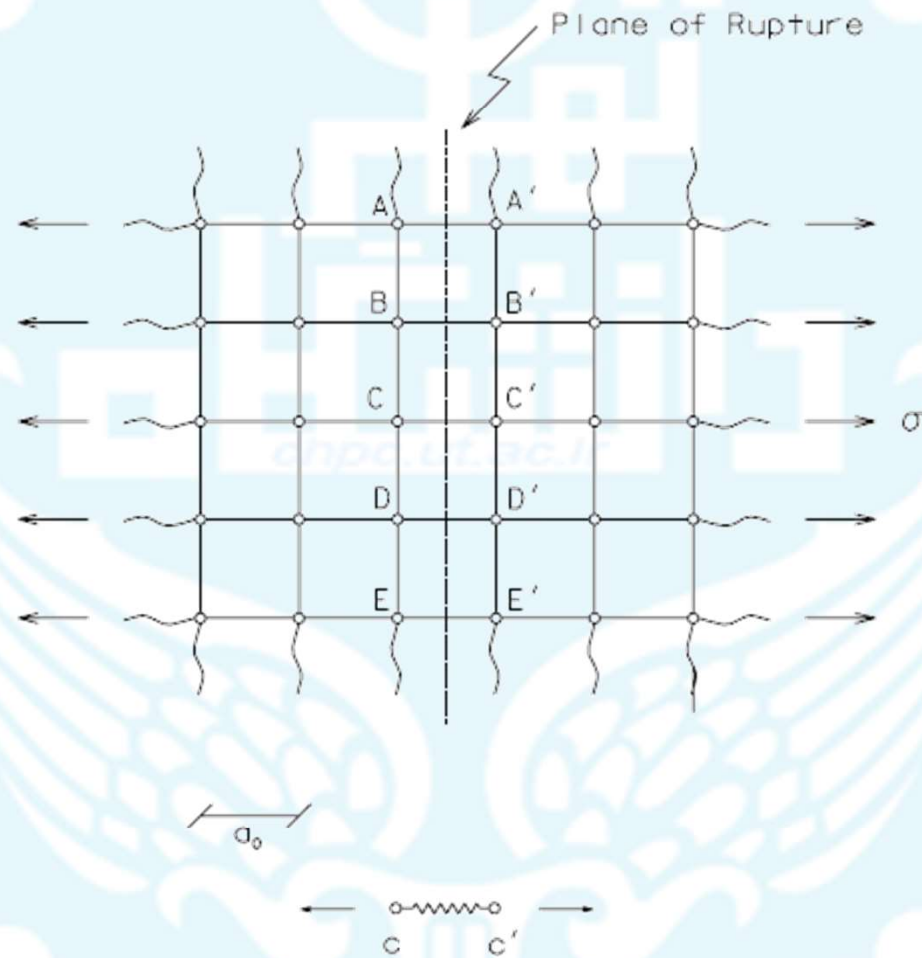
$$\gamma_s = 0.077 \text{ J / m}$$

Most metals:

$$\gamma_s = 1 \text{ J / m}$$

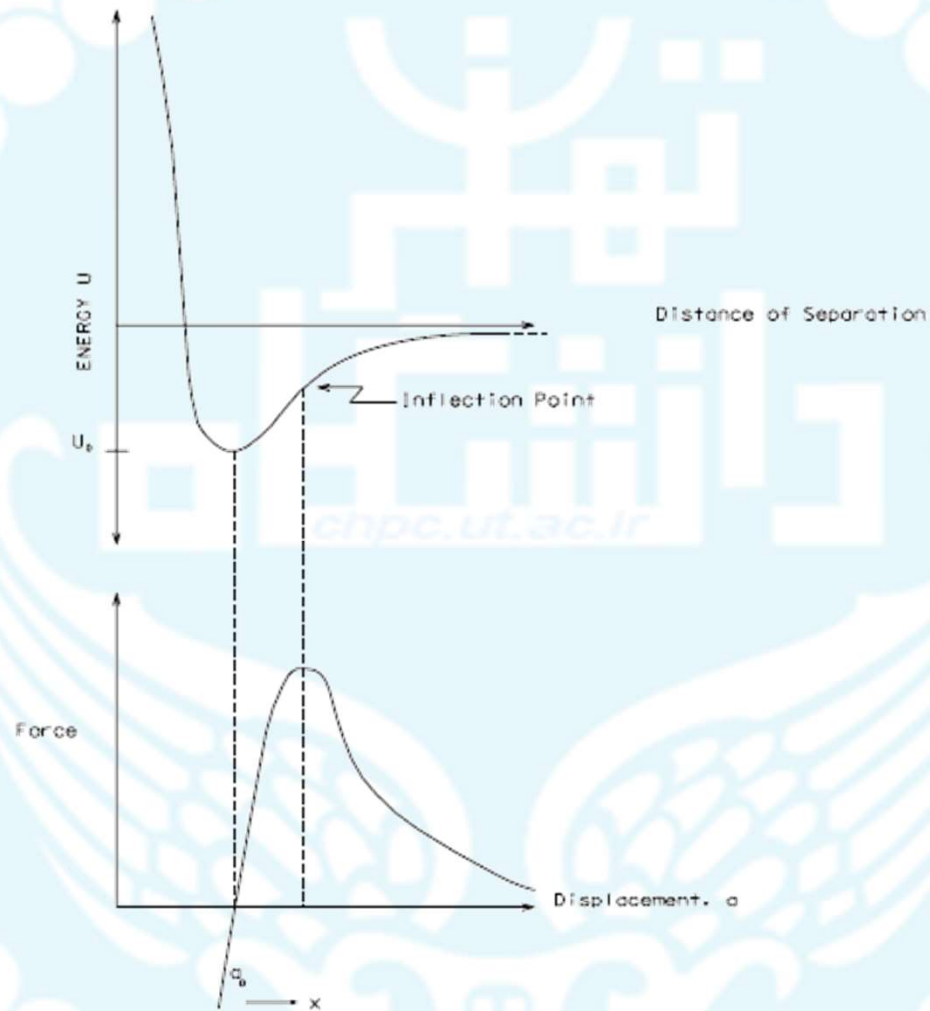


Assume a bond is uniformly broken between two layers of atoms:



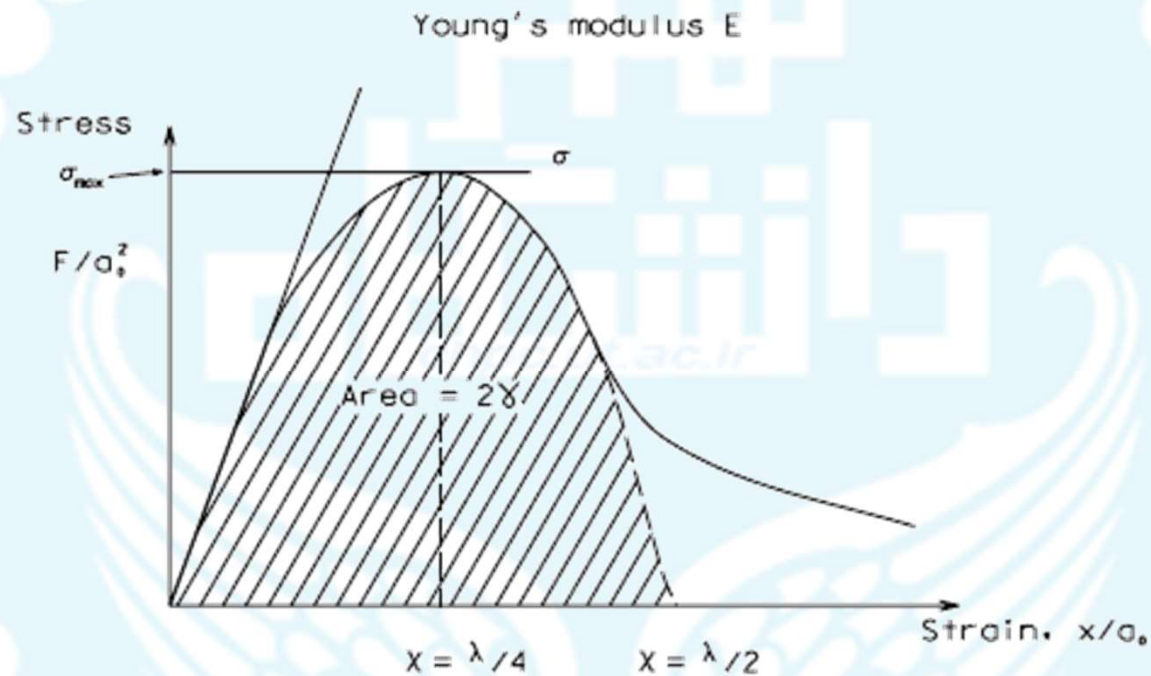
Uniformly Stressed Layer of Atoms Separated by a_0

and from typical energy and binding force of two atoms:



Energy and Force Binding Two Adjacent Atoms

a simplified force-displacement (stress-strain) variation may be assumed:



Stress Strain Relation at the Atomic Level

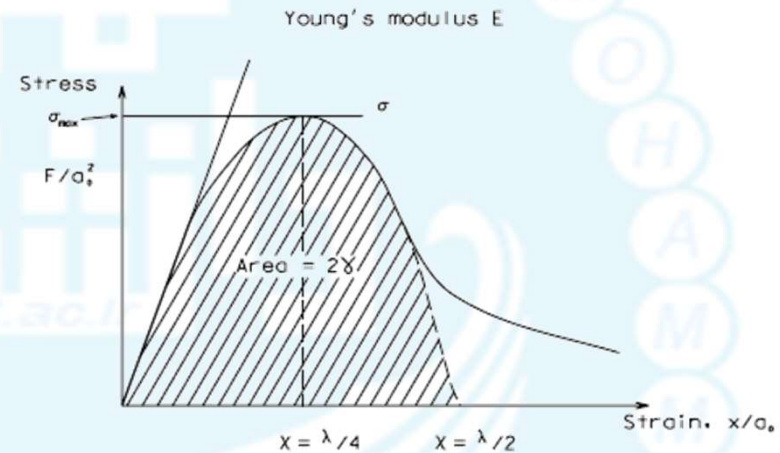
a simplified force-displacement (stress-strain) variation may be assumed:

$$x = a - a_0$$

$$\varepsilon = \frac{x}{a_0}$$

$$\sigma = \frac{F}{a_0^2}$$

$$\sigma = \sigma_{\max}^t \sin\left(2\pi \frac{x}{\lambda}\right)$$



Stress Strain Relation at the Atomic Level

The maximum stress would occur at $x = \frac{\lambda}{4}$

The energy required to separate two atoms is given by the area under the sine curve:

$$U = \int_0^{\lambda/2} \sigma_{\max}^t \sin\left(2\pi \frac{x}{\lambda}\right) dx = \dots = \frac{\lambda}{\pi} \sigma_{\max}^t$$

which must be equal to the surface energy $2\gamma_s$

$$U = 2\gamma_s \rightarrow \lambda = \frac{2\gamma_s \pi}{\sigma_{\max}^t}$$

For very small displacements:

$$\sigma = \sigma_{\max}^t \sin\left(2\pi \frac{x}{\lambda}\right) \approx \sigma_{\max}^t \left(2\pi \frac{x}{\lambda}\right)$$

Alternatively, based on the elastic Hook's law:

$$\sigma = E\varepsilon = E \frac{x}{a_0}$$

Eliminating x ,

$$\sigma_{\max}^t \approx \frac{E}{a_0} \frac{\lambda}{2\pi}$$

and from $\lambda = \frac{2\gamma_s \pi}{\sigma_{\max}^t}$

$$\sigma_{\max}^t \approx \sqrt{\frac{E\gamma_s}{a_0}}$$

For a typical steel, it results to unrealistic ideal strength:

$$\left. \begin{array}{l} E = 2 \times 10^{11} \text{ N / m}^2 \\ \gamma_s = 1 \text{ J / m}^2 \\ a_0 \approx 2 \times 10^{-10} \text{ m} \end{array} \right\} \rightarrow \sigma_{\max}^t = 3.16 \times 10^{11} \text{ N / m}^2 \left(\approx \frac{E}{6} \right)$$

If it is further assumed that: $a \approx \frac{\lambda}{2}$

$$\sigma_{\max}^t \approx \frac{E}{a_0} \frac{\lambda}{2\pi} = \frac{E}{a_0} \frac{a}{\pi}$$

$$\lambda = 2a = \frac{2\gamma_s \pi}{\sigma_{\max}^t}$$

leads to:

$$\gamma_s \approx \frac{E}{a_0} \left(\frac{a}{\pi}\right)^2$$

which can further be simplified to:

$$a \approx a_0 \rightarrow \gamma_s \approx \frac{E a_0}{10} \xrightarrow{\sigma_{\max}^t \approx \sqrt{\frac{E \gamma_s}{a_0}}} \sigma_{\max}^t \approx \frac{E}{\sqrt{10}}$$

Griffith Theory:

Assuming an elliptical imperfection,

$$\sigma_{\max}^t = \sigma^{\text{applied}} \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

Furthermore:

$$\rho \approx a_0$$

$$\sigma_{\max}^t = \sigma^{\text{applied}} \left(1 + 2\sqrt{\frac{a}{a_0}}\right) \xrightarrow{a_0 \ll a} \sigma_{\max}^t \approx 2\sigma^{\text{applied}} \sqrt{\frac{a}{a_0}}$$

$$\sigma_{\max}^t \approx \underbrace{2\sigma^{\text{applied}} \sqrt{\frac{a}{a_0}}}_{\text{macro}} = \underbrace{\sqrt{\frac{E\gamma_s}{a_0}}}_{\text{micro}}$$

$$\sigma_{\max}^t \approx \underbrace{2\sigma^{applied}}_{macro} \underbrace{\sqrt{\frac{a}{a_0}}}_{micro} = \underbrace{\sqrt{\frac{E\gamma_s}{a_0}}}_{micro}$$

Griffith method managed to correlate a macro solution, based on linear elastic solution of Inglis, with the micro solution, based on interatomic relations.

Finally, the maximum bearable macroscopic applied stress can be computed:

$$\sigma^{applied} = \sqrt{\frac{E\gamma_s}{4a}}$$

$$\sigma^{applied} = \sqrt{\frac{E\gamma_s}{4a}}$$

For instance, for a more realistic flaw size a :

$$2a \approx 5000a_0 \rightarrow \sigma^{applied} = \frac{E}{100\sqrt{10}}$$

and a much better prediction is obtained for steel:

$$\sigma^{applied} = \frac{E}{2000}$$

The background features a large, light blue watermark of the Shahrood University of Technology logo. The logo is circular with a scalloped border. Inside the circle, there is a stylized emblem at the top, followed by the university's name in Persian script, and then the name in English, 'SHAHROOD UNIVERSITY OF TECHNOLOGY', arranged in a circular path. The text 'Griffith Thermodynamics' is centered over the upper part of the logo.

Griffith Thermodynamics

and

Fracture Energy

Consider a crack in a deformable continuum subjected to arbitrary loading. The first law of thermodynamics states that:

$$\frac{d}{dt}(U_K + U_s + U_\Gamma) = \frac{d}{dt}(W + Q)$$

U_K the kinematic energy

U_s the total internal strain energy

U_Γ the surface energy

W the external work

Q the heat input to the system

The first law of thermodynamics for an adiabatic quasi-static system states that the change in total energy is proportional to the amount of performed work:

$$\frac{d}{dt}(U_s + U_\Gamma) = \frac{d}{dt}(W)$$

U_s the total internal strain energy

U_Γ the surface energy

W the external work

or in terms of **virtual crack extension** da

$$\frac{d}{da}(U_s + U_\Gamma) = \frac{d}{da}(W) \quad \text{or} \quad \frac{\partial W}{\partial a} = \frac{\partial U_s}{\partial a} + \frac{\partial U_\Gamma}{\partial a}$$

This equation represents the energy balance during crack growth, and indicates that the work rate supplied to the continuum by the applied external load is equal to the surface energy dissipated during crack propagation, U_Γ , plus the rate of strain energy, U_s , decomposed into elastic U_s^e and plastic U_s^p parts

$$U_s = U_s^e + U_s^p$$

in terms of the potential energy, Π

$$\Pi = U_s^e - W$$

$$-\frac{\partial \Pi}{\partial a} = -\frac{\partial U_s^e}{\partial a} + \frac{\partial W}{\partial a} = -\frac{\partial U_s^e}{\partial a} + \frac{\partial U_s}{\partial a} + \frac{\partial U_\Gamma}{\partial a} = \frac{\partial U_s^p}{\partial a} + \frac{\partial U_\Gamma}{\partial a}$$

Therefore, the energy available for crack growth is compared with the resistance of the material that must be overcome for crack growth. It also indicates that the decrease rate of potential energy during crack growth is equal to the rate of energy dissipated in plastic deformation and crack growth.

For a perfectly brittle material, U_s^p vanishes and

$$-\frac{\partial \Pi}{\partial a} = \frac{\partial W}{\partial a} - \frac{\partial U_s^e}{\partial a} = \frac{\partial U_\Gamma}{\partial a} = 2\gamma_s$$

γ_s is the surface energy and the factor 2 represents the existence of two material surfaces upon fracture. The value of γ_s is experimentally measured for different materials.

$$\gamma_s = 0.077 \quad \text{water}$$

$$\gamma_s = 1 \quad \text{most metals}$$

Therefore,

$$\frac{\partial}{\partial a} (\Pi + U_\Gamma) = 0$$

The main equation for brittle materials:

$$-\frac{\partial \Pi}{\partial a} = \frac{\partial W}{\partial a} - \frac{\partial U_s^e}{\partial a} = \frac{\partial U_\Gamma}{\partial a} = 2\gamma_s$$

is defined as the Griffith crack growth energy, G ,

$$G = -\frac{\partial \Pi}{\partial a} = 2\gamma_s$$

Thus, a criterion for crack propagation can be expressed as an inequality between the energy release rate per unit crack extension and the surface energy:

$$\frac{d\Pi}{da} \geq 2\gamma_s \quad \text{or} \quad G \geq G_c$$

A crack's growth can be considered unstable or stable when the energy at equilibrium is a maximum or minimum, respectively. Therefore, a sufficient condition for crack stability is obtained from the second derivative of $(\Pi + U_{\Gamma})$

$$\frac{\partial}{\partial a}(\Pi + U_{\Gamma}) = 0$$

$$\frac{\partial^2(\Pi + U_{\Gamma})}{\partial a^2} \begin{cases} < 0 & \text{unstable fracture} \\ > 0 & \text{stable fracture} \\ = 0 & \text{neutral equilibrium} \end{cases}$$

Griffith has shown that for a plane stress infinite plate with a central crack of length $2a$ and unit thickness subjected to unilateral tensile loading, the strain energy required to introduce the crack is equal in magnitude to the work required to close the crack by the stresses acting on it. Using the solutions for displacements of the free surface of one of the crack faces

$$\Pi = 4 \int_0^a \frac{1}{2} \sigma u_y(x) dx = \frac{\pi \sigma^2 a^2}{2E'}$$

$$E' = \begin{cases} E & \text{plane stress} \\ \frac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

$$G = -\frac{\partial \Pi}{\partial a} = -\frac{\pi \sigma^2 a}{E'}$$

and from definition of K_I

$$K_I = \sigma \sqrt{\pi a}$$

It is found that:

$$G = \frac{K_I^2}{E'}$$

So, the crack stability criterion may be written as:

$$G \leq G_c$$

or

$$K_I \leq K_c$$

The critical stress for cracking is denoted by σ_{cr}

$$\sigma_{cr} = \sqrt{\frac{2E'\gamma_s}{\pi a}}$$

The critical stress intensity factor K_c for the infinite tensile plate with a central crack can then be defined as:

$$K_c = \sigma_{cr} \sqrt{\pi a}$$

and an unstable crack extension occurs if $K_I = K_c$.

The Griffith fracture energy method:

- Constitutes no simplification
- Based on global laws of thermodynamics
- Can be used for any type of fracture problems (dynamic, non-homogeneous, multi-physics, nonlinear, large deformations, ...)
- Non-local solution
- Closely related to SIF for LEFM problems
- No need for complex elasticity solutions
- Physical interpretation and unit

In order to extend the Griffith model further to **quasi-brittle materials**, the following modifications can be made:

$$\frac{\partial U_{\Gamma}}{\partial a} = 2(\gamma_s^e + \gamma_s^p)$$

For ductile metals

$$\gamma_s^p \gg \gamma_s^e$$

or alternatively in terms of a correction factor α_p

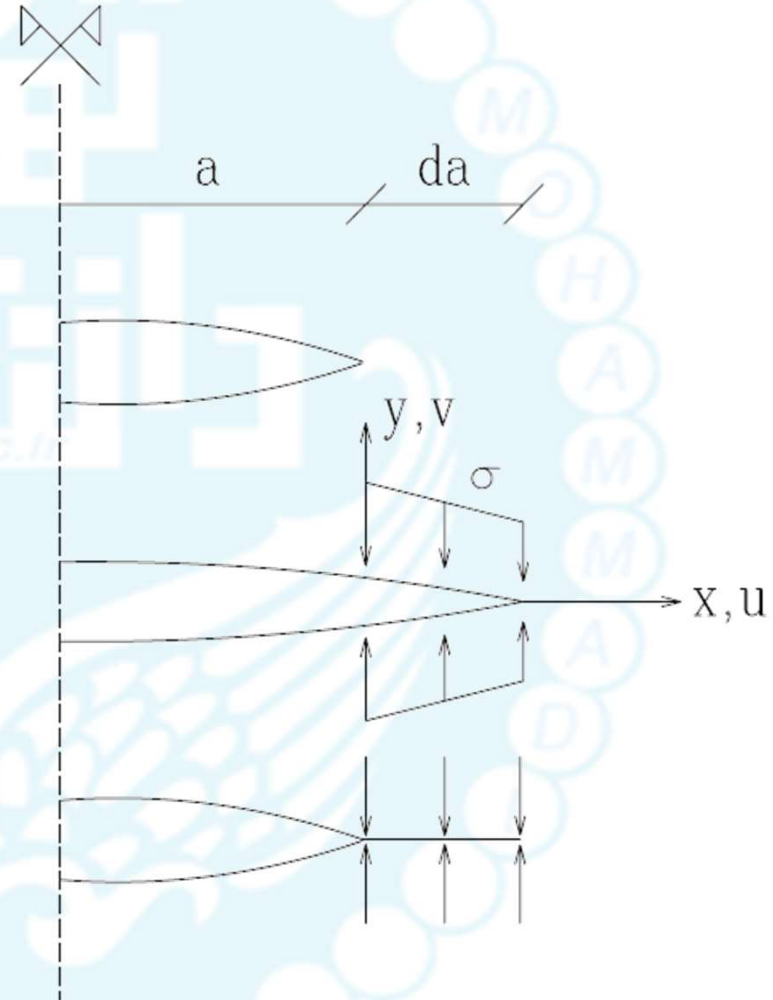
$$\sigma_{cr} = \sqrt{\frac{2E' \alpha_p \gamma_s}{\pi a}}$$

Equivalence of G and K in LEFM:

Virtual crack extension:

the energy released during a colinear unit crack extension can be determined by calculating the work done by the surface forces acting across the length da when the crack is closed from length $(a+da)$ to length a .

$$G = \frac{2}{\Delta a} \int_a^{a+\Delta a} \frac{1}{2} \sigma_{yy} v dx$$



Using the Westergaard solutions for displacement and stress states,

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos \theta \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right]$$

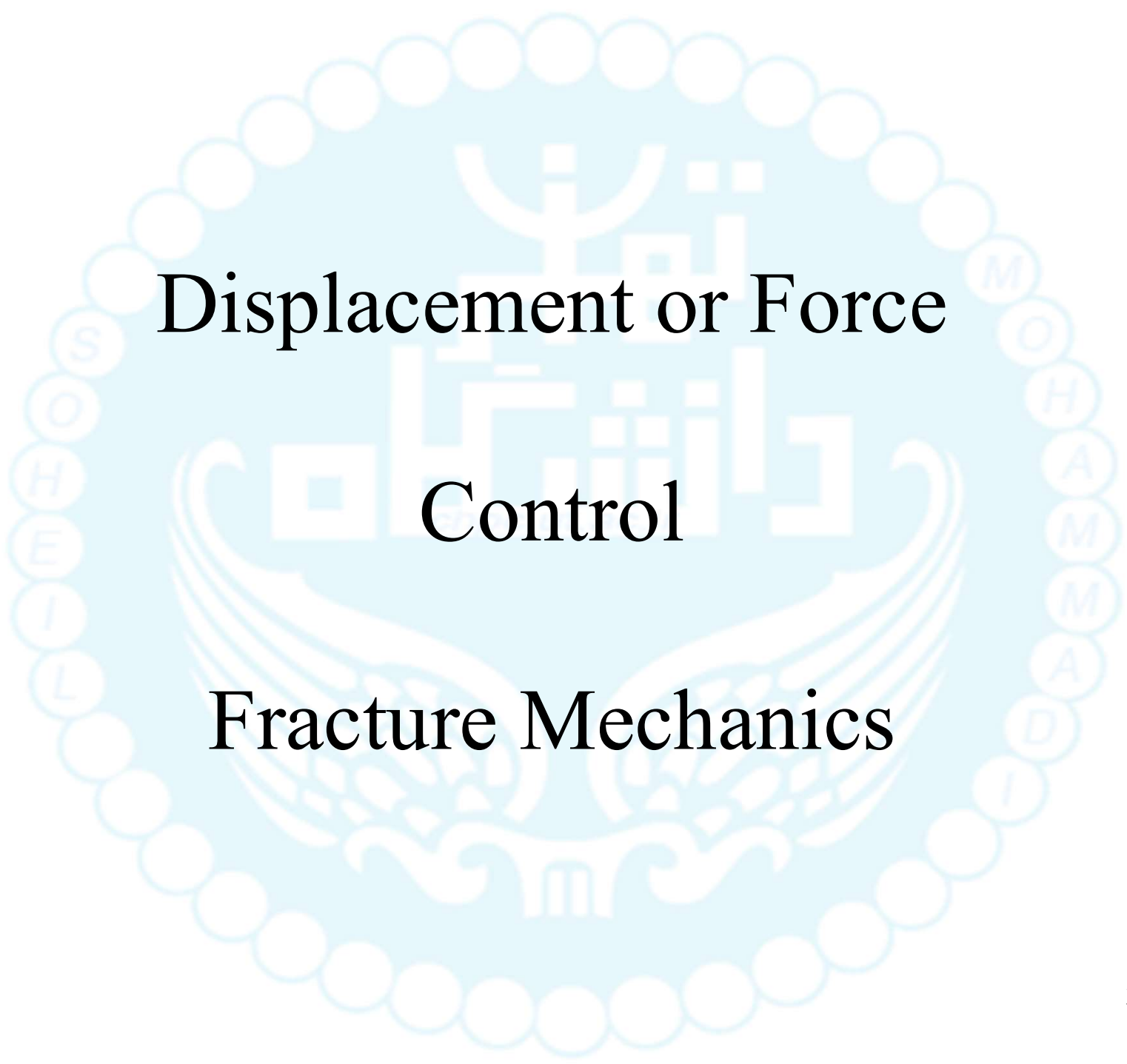
$$v = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[\kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right]$$

leads to:

$$G = \frac{K_I^2}{E'}$$

$$E' = E \quad \text{plane stress}$$

$$E' = \frac{E}{1 - \nu^2} \quad \text{plane strain}$$



Displacement or Force

Control

Fracture Mechanics

In many load control problems, as soon as K reaches to its critical value of K_c , the crack extends, which leads to larger value of K and further crack extension, and a final failure.



$$K_I = 2\sqrt{3} \frac{Pa}{c^{1.5}}$$

double cantilever beam

This, however, may not be the case for some displacement control problems, as the deformation of the whole specimen is constrained by the applied displacement control pattern, even though it experiences larger crack lengths (due to stable crack propagation).

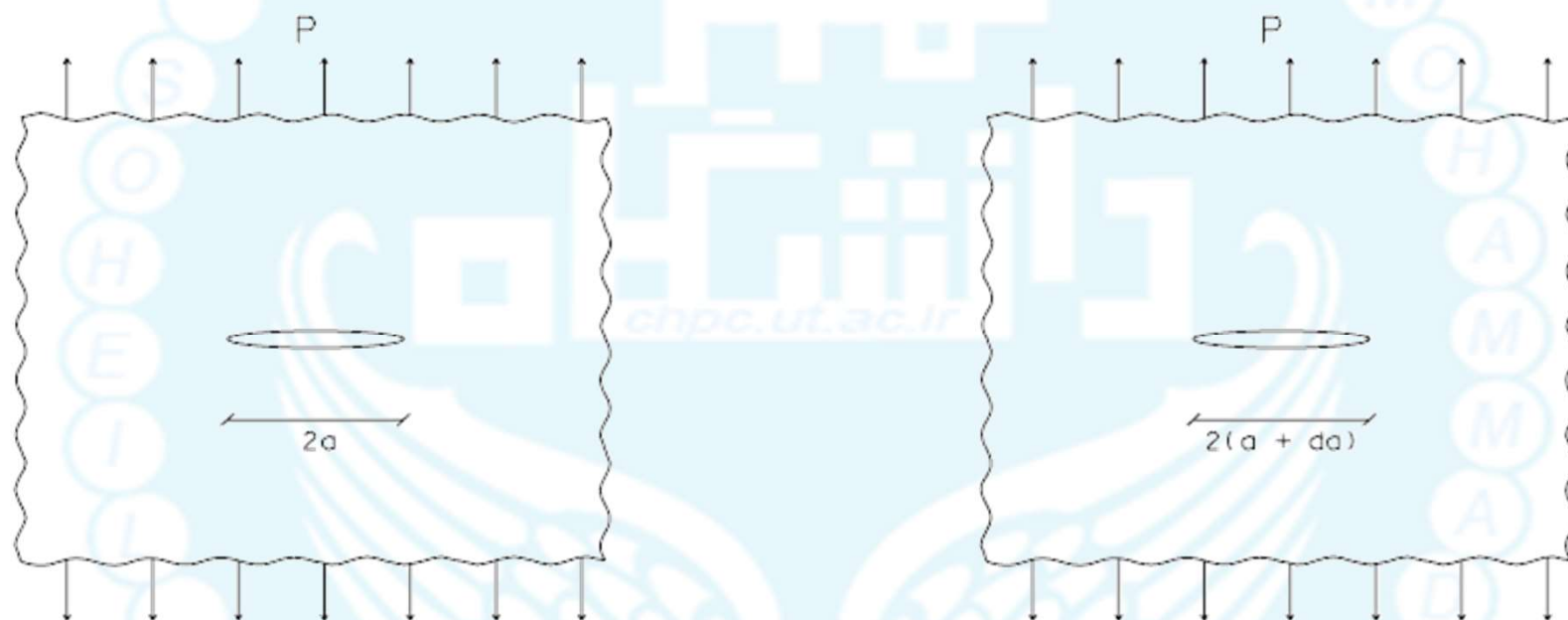
Displacement control versus load control

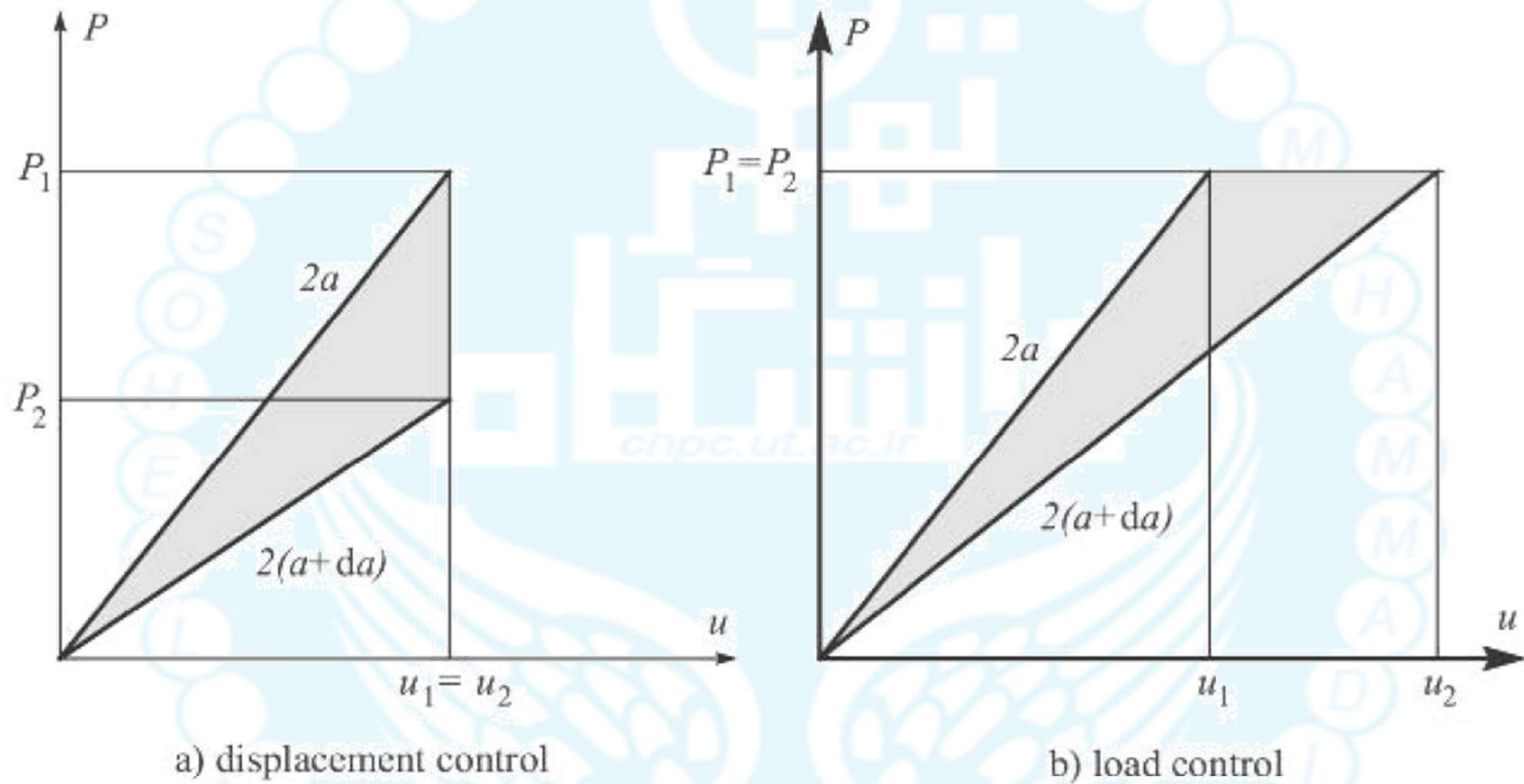
Fixed grip versus fixed load

Two limiting cases of the fixed grip (displacement control) and the fixed load (load control) are analyzed.

An infinite plate with a crack of length $2a$ subjected to the loading P is now considered.

The change in potential energy is determined as the crack length is increased to $2(a + da)$.





Load–displacement curves for displacement control and load control problems.

In the first case ($u_2 = u_1$), the external work is zero ($\Delta W = 0$) and any increase in crack length results in a decrease in stored elastic strain energy:

$$\Delta U = \frac{1}{2} P_2 u_1 - \frac{1}{2} P_1 u_1 < 0$$

$$\Pi_2 - \Pi_1 = \Delta W - \Delta U = \frac{1}{2} (P_1 - P_2) u_1$$

This release of excess energy is consumed to form the surface energy.

Under the fixed loading condition ($P_2 = P_1$), neither the external work nor the release of internal strain energy is zero,

$$\Delta W = P_1(u_2 - u_1)$$

$$\begin{aligned}\Pi_2 - \Pi_1 &= \Delta W - \Delta U = P_1(u_2 - u_1) - \frac{1}{2} P_1(u_2 - u_1) \\ &= \frac{1}{2} P_1(u_2 - u_1)\end{aligned}$$

Half of the produced external work is consumed into strain energy, and the other half is released to form surface energy.

Both cases may be presented in a unified formulation, by defining the elastic compliance matrix C :

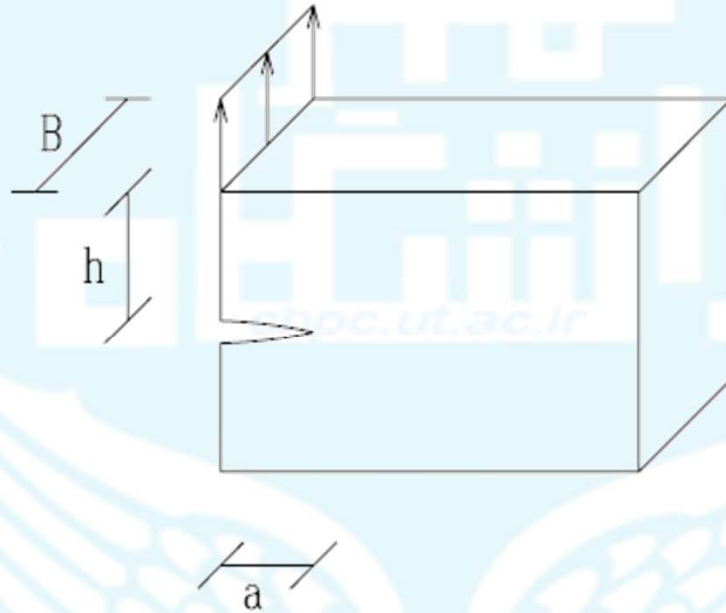
$$u = CP$$

$$d\Pi = \frac{1}{2} CP dP \rightarrow \begin{cases} d\Pi = \frac{1}{2} u dP & \text{for fixed displacement} \\ d\Pi = \frac{1}{2} P du & \text{for fixed loading} \end{cases}$$

and more generally,

$$d\Pi = \frac{1}{2} P d(CP)$$

As an example, let us consider the double cantilever beam problem (fixed loading).



K_I for DCB using the Compliance Method

From strength of materials:

$$C = \underbrace{\frac{24}{EB} \int_0^a \frac{x^2}{h^3} dx}_{flexural} + \underbrace{\frac{6(1+\nu)}{EB} \int_0^a \frac{1}{h} dx}_{shear}$$

for $\nu = \frac{1}{3}$

$$C = \frac{8}{EB} \int_0^a \left(\frac{3x^2}{h^3} + \frac{1}{h} \right) dx$$

$$\frac{dC}{da} = \frac{8}{EB} \left(\frac{3a^2}{h^3} + \frac{1}{h} \right)$$

For a fixed loading problem:

$$d\Pi = \frac{1}{2} P du \rightarrow G = \frac{d\Pi}{da} = \frac{1}{2} P \frac{d(CP)}{da} = \frac{1}{2} P^2 \frac{dC}{da}$$

Considering the thickness of the specimen, B ,:

$$G = \frac{1}{2} \frac{P^2}{B} \frac{dC}{da} = \frac{4^2}{EB^2h^3} (3a^2 + h^2)$$

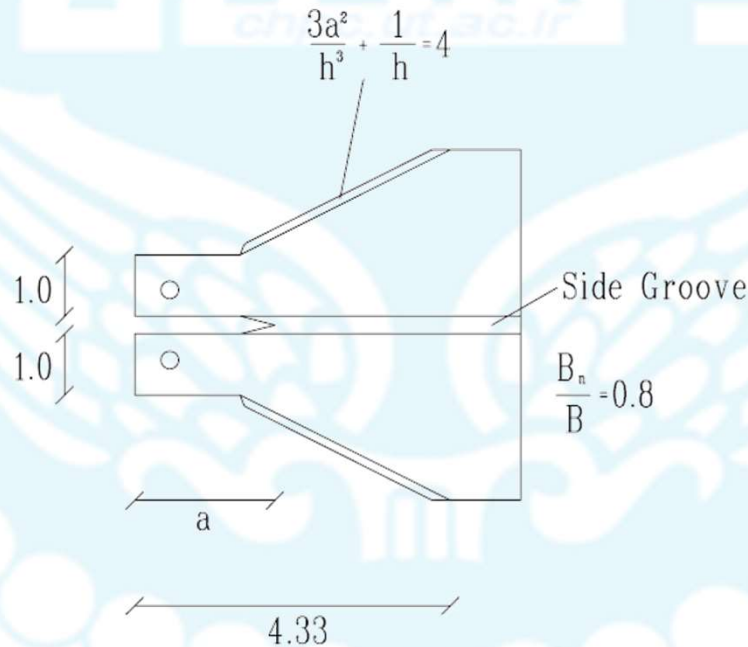
$$K_I^2 = GE \rightarrow K_I = \frac{2P}{B} \left(\frac{3a^2}{h^3} + \frac{1}{h} \right)^{\frac{1}{2}}$$

K_I increases with a , which leads to an unstable crack growth.

To avoid unstable crack growth, one may design a specimen with constant:

$$\frac{3a^2}{h^3} + \frac{1}{h} = \text{constant}$$

as proposed by Mostovoy (1967).



Variable Depth Double Cantilever Beam